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BEM SOLUTION OF TWO-DIMENSIONAL CONTACT PROBLEMS BY WEAK APPLICATION OF CONTACT CONDITIONS WITH NON-CONFORMING DISCRETIZATIONS

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Abstract—Non-conforming discretizations of the surfaces involved in the contact problem are sometimes required, due to the geometry and/or the load. A strong application of the contact conditions directly relating variables (displacements and tractions) of the nodes of the discretizations, a general approach called by the authors node-to-point contact scheme, may lead to unsatisfactory results. In this paper, a weak application of the contact conditions, by means of the principle of virtual work, is developed for boundary integral equations. The formulation is presented for two-dimensional problems without or with friction, using the Coulomb model. The modelization is made using the Boundary Element Method and the problem is solved with an incremental procedure based on a displacement scaling approach. The solution scheme proposed is applicable to any contact problem (with small or large displacements) and is validated in this paper by applying it to receding, conforming and advancing contact problems, the jumps in the contact stresses that appeared in node-to-point contact schemes, not having been found in the problems tested. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Knowledge of the contact problem started with the studies of Hertz (1896) in 1882, who determined the distribution of pressures throughout the contact zone that appear when two bodies with curved surfaces are pressed against each other. Many problems, although usually involving simple geometries with infinite dimensions and frictionless character, have been analytically solved since then. Many of these problems can be found in Gladwell (1980) and Johnson (1985).

Complicated geometries or loads require the use of numerical methods. The references to Chan and Tuba (1971), Fredriksson (1976), Okamoto and Nakazawa (1979), Oden and Pires (1984), Bathe and Chaudhary (1985) and Klarbring and Björkman (1992) are, among many others, representative of those using the Finite Element Method (FEM) as numerical technique.

With reference to the Boundary Element Method (BEM), the earliest contributions were due to Andersson *et al.* (1980, 1982) and París and Garrido (1985) who later analysed different aspects of the problem (París and Garrido, 1988, 1989; Garrido *et al.*, 1991) and extended the formulation to the three-dimensional case (París *et al.*, 1994; Garrido *et al.*, 1991; Foces *et al.*, 1993). Other approaches were presented by Takahashi and Brebbia (1988) using a flexibility approach and Man *et al.* (1992) using a load scaling procedure.

All these algorithms based on BEM have in common the requirement of identical discretizations of the two surfaces involved in the contact, with elements candidates for contact of the same length, so that the contact conditions are applied between corresponding nodes that originally (conforming or receding problems) or during the application of the load (advancing problems) are assumed to occupy a common position.

There are situations where it would be advisable, even necessary, to have algorithms allowing different discretizations of the two surfaces involved in the contact problem. One of these situations is clearly the case of assumption of rigid body for one of the solids

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involved in the problem (e.g. the case of contact between some fibres and matrices in composite materials). Another situation would be the case of contact between surfaces of different curvatures (thus avoiding the tedious work of discretizing the two surfaces in an identical manner). A third case, not as obvious but probably of greater interest, would be the case of appearance of moderate relative displacements, in such a way that even original conforming discretizations may require during the application of the load the use of a non-conforming approach to avoid jumps of the results originated by the conforming character of the aforementioned approaches. A fourth situation would be the case of quasistatic analysis of dynamics problems, avoiding the need for a remeshing procedure for each particular configuration considered.

In the above mentioned works of Bathe and Chaudhary (1985) (2-D and axisymmetric) and Klarbring and Björkman (1992) (3-D) there are, using Finite Elements, algorithms to deal with non-conforming discretizations.

Blázquez *et al.* (1992), París *et al.* (1995), Olukoko *et al.* (1993) and Huesmann and Kuhn (1994) presented approaches based on BEM to deal with non-conforming discretizations. In these three approaches there is a common philosophy of strong application of contact conditions: displacements and tractions of the nodes of the contact zone are related in some way in accordance with a node-to-point contact scheme. Thus, equilibrium and compatibility relations are directly forced between the nodes of one body and the corresponding contacting points (not necessarily nodes) of the other body.

Blázquez *et al.* (1997) compared all these existing proposals based on this node-to-point approach, finding no serious differences among them. To obtain good results they showed that displacements must be better represented than tractions, even in the presence of singular stresses. Nevertheless, the common problem with these approaches is that some jumps in the tractions along the contact zone may take place in some cases when, as a result of the discretization performed, there is a poor definition of the displacements at the nodes of the body that controls the stresses, the nodes being forced to occupy positions different from those they would naturally tend to occupy. This problem is also treated in detail in Blázquez and París (1997), with some advice on how to avoid the problem.

It seems, speaking in general terms, that a node-to-point contact scheme may be too strict to force contact conditions between two non-conforming discretizations. In this paper a different approach based on a weak application of the contact conditions, by means of the application of the Principle of Virtual Work, is going to be followed. A similar scheme is followed by Schnack (1987) in coupling Boundary and Finite Element Methods.

First of all, a brief review of the contact problem and the Boundary Element Method is given in Sections 2 and 3, respectively. The description of the weak application of contact conditions is made in Section 4. The algorithm of solution, which is not substantially altered by the manner of application of the contact conditions, is described in Section 5. Finally, the classical problems of receding, conforming and advancing contact problems are considered in Section 6 to check the disappearance of the jumps that arose in the node-to-point contact schemes. The problems considered in this paper are restricted to those able to be modelled two-dimensionally.

2. THE CONTACT PROBLEM

Let us assume, Fig. 1, two bodies A and B, occupying the domains D^A and D^B with boundaries ∂D^A and ∂D^B, which interact between them through a common contact zone $\partial D_c^A = \partial D_c^B = \partial D_c$. The loads, which are assumed to depend on a parameter λ, are given by the tractions and displacements prescribed along the boundaries ∂D_t^K, ∂D_u^K and ∂D_{ut}^K, respectively, $\partial D_t^K + \partial D_u^K + \partial D_{ut}^K = \partial D_l^K = \partial D^K - \partial D_c^K$, $K = A, B$. Thus, the boundary conditions of the problem along ∂D_l^K are expressed as:

$$u_i^K = \bar{u}_i^K(\lambda) \quad \text{along } \partial D_u^K; \quad i = 1,2; \quad K = A, B$$

$$t_i^K = \bar{t}_i^K(\lambda) \quad \text{along } \partial D_t^K; \quad i = 1,2; \quad K = A, B$$

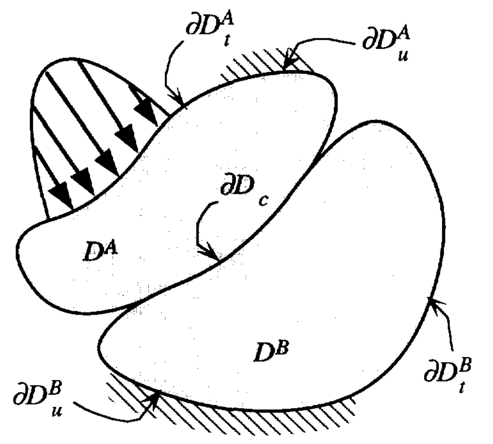


Fig. 1. The contact problem.

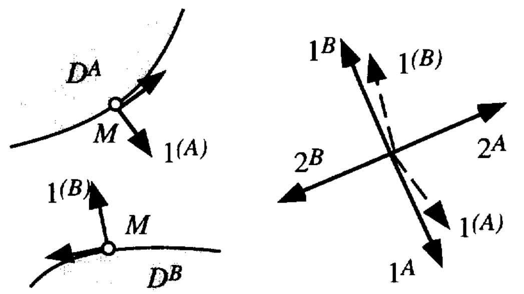


Fig. 2. The contact coordinate system.

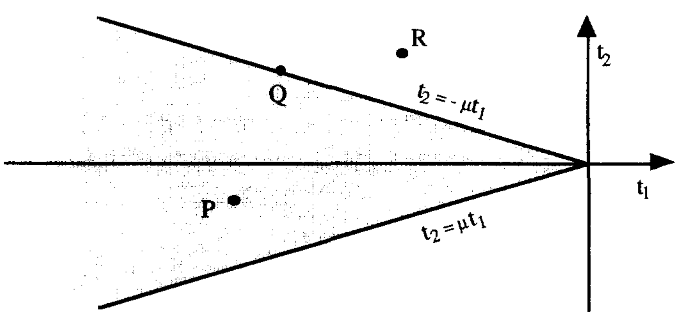


Fig. 3. Graphical representation of Coulomb friction law.

$$u_i^K = \bar{u}_i^K(\lambda), \quad t_j^K = \bar{t}_j^K(\lambda) \quad \text{along } \partial D_{ut}^K, \quad i,j = 1,2; \quad i \neq j; \quad K = A, B \tag{1}$$

Contact conditions are established in a common coordinate system, Fig. 2. At each point M of the contact zone, direction 1 is taken as the average of the two outward normals to the boundaries, direction 2 being perpendicular to direction 1 and anti-clockwise.

Coulomb friction law, represented in Fig. 3, has been assumed in this work. A point M of the contact zone whose stress state is represented by a situation such as that denoted by P is in adhesion, not admitting relative tangential displacements. A point M of the contact zone whose stress state is represented by a situation such as that denoted by Q admits relative tangential displacements. Stress states like that represented by R are not admissible in the Coulomb model. Thus, the contact zone is divided into two subzones: subzone of adhesion $\partial D^A_{ca} = \partial D^B_{ca} = \partial D_{ca}$ and sliding subzone $\partial D^A_{cd} = \partial D^B_{cd} = \partial D_{cd}$.

The contact conditions are classified in three groups: equilibrium, compatibility of normal displacements and those derived from the friction law considered. These conditions, in the coordinate system defined by Fig. 2, are:

• Equilibrium:

$$t_i^A(M) = t_i^B(M), \quad i = 1,2 \tag{2}$$

• Compatibility of normal displacements:

$$u_1^A(M) + u_1^B(M) = 0 \tag{3}$$

• Friction law:

$$\begin{aligned} M \in \partial D_{ca} &\Rightarrow u_2^A(M) + u_2^B(M) = 0 \\ M \in \partial D_{cd} &\Rightarrow t_2^K(M) = \pm \mu t_1^K(M), \quad k = A, B \end{aligned} \tag{4}$$

The limit of application of these conditions is given by:

• Contact pressures:

$$t_1^K(M) \leqslant 0, \quad K = A, B \tag{5}$$

• Maximum admissible tangential stresses:

$$|t_2^K(M)| \leqslant \mu |t_1^K(M)|, \quad K = A, B \tag{6}$$

• Dissipative character of friction:

$$t_2^K(M)(u_2^A(M) + u_2^B(M)) \leqslant 0, \quad K = A,B \tag{7}$$

3. THE BOUNDARY ELEMENT METHOD

In absence of body forces, boundary integral formulation of elastic bodies can be expressed in terms of Somigliana Identity, París and Cañas (1997), which with reference to a point x of the boundary of body K ($K = A, B$) reads:

$$C_{ji}^K(x)\,\Delta u_i^K(x) + \int_{\partial D^K} T_{ji}^{\psi}(x,y)\,\Delta u_i^K(y)\,\mathrm{d}s(y) = \int_{\partial D^K} \Psi_{ji}(x,y)\,\Delta t_i^K(y)\,\mathrm{d}s(y) \tag{8}$$

where $K = A, B$; x, $y \in \partial D^K$; Ψ_{ji} and T_{ji}^{ψ} are, respectively, displacements and tractions associated to Kelvin fundamental solution. The term C_{ji}^K is usually known as the coefficient matrix of the free term, its value depending on the local geometry at point x where the integral equation is applied.

The Identity has been written in incremental form to deal with the non linearities that may arise during the application of the load. This equation can be applied to any point of the boundaries of the bodies A and B, usually the nodes of the discretization, in such a way that these equations, together with the boundary conditions of the contact-free zone [eqn (1)] and the contact conditions [equations (2)–(4) with the limitations (5)–(7)], allow the problem to be solved.

The Boundary Element Method consists of the replacement of the boundary ∂D by a set of elements along which displacements and tractions vary in a certain way. In this paper all the results presented are obtained with straight linear elements allowing discontinuities of the stress vector at the junction of the elements, París and Cañas (1997). Using quadratic elements, a contact procedure designed node by node may lead, when the central node of the element contacts, to overlapping in the contacted zone of the element and tractions in the non contacted zone of the element, Andersson and Allan-Persson (1983). On the other hand, a contact procedure conceived element by element would imply considering a non linear step as a linear one, leading to an oscillatory solution in the contact pressures, Man *et al.* (1993). To avoid these problems, particular decisions should be taken in order to use quadratic elements, whereas no special precautions are required using linear elements, the accuracy being satisfactory.

Displacements and tractions along an element k of body K, ∂D_k^K, are expressed as a function of the nodal values by means of the corresponding shape functions:

$$\Delta \mathbf{u}^{\partial D_k^K}(\xi) = \mathbf{N}^{kK}(\xi)\,\Delta \mathbf{u}^{kK} \tag{9}$$

$$\Delta \mathbf{t}^{\partial D_k^K}(\xi) = \mathbf{N}^{kK}(\xi)\,\Delta \mathbf{t}^{kK} \tag{10}$$

where ξ is a natural coordinate defined along ∂D_k^K, $\Delta \mathbf{u}^{\partial D_k^K}(\xi)$ and $\Delta \mathbf{t}^{\partial D_k^K}(\xi)$ are, respectively, the increments of displacements and tractions vectors at the point of the element of coordinate ξ, $\mathbf{N}^{kK}(\xi)$ is a matrix containing the shape functions, and $\Delta \mathbf{u}^{kK}$ and $\Delta \mathbf{t}^{kK}$ are vectors that include the displacements and tractions of the nodes of the element k of solid K.

Performing the discretization procedure in eqn (8) and applying it to the nodes of the discretization (taken as usual as collocation points) of the boundaries of the two bodies leads to the following system of equations, where for the sake of simplicity mixed conditions have not been explicitly represented:

$$[\mathbf{H}_t^K \quad \mathbf{H}_u^K \quad \mathbf{H}_c^K]\begin{Bmatrix}\Delta \mathbf{u}_t^K \\ \Delta \bar{\mathbf{u}}^K \\ \Delta \mathbf{u}_c^K\end{Bmatrix} = [\mathbf{G}_t^K \quad \mathbf{G}_u^K \quad \mathbf{G}_c^K]\begin{Bmatrix}\Delta \bar{\mathbf{t}}^K \\ \Delta \mathbf{t}_u^K \\ \Delta \mathbf{t}_c^K\end{Bmatrix} \tag{11}$$

These $\mathbf{H}$ are matrices whose coefficients represent integrations of T_{ji}^{ψ} along the elements of body K plus the value of the free term, C_{ji}^K, when appropriate. Correspondingly these $\mathbf{G}$ are matrices whose coefficients represent integrations of Ψ_{ji}. The subindexes t, u, c make reference to the zone of the boundary along which the integration is being performed. In system (11), boundary conditions (1) have already been imposed along ∂D_t^K.

The casuistry of application of the direct boundary conditions expressed by (1) is explained in detail in París and Cañas (1997). A new way to impose contact conditions (2)–(4) on the discretized bodies will be described in the next Section.

4. WEAK IMPOSITION OF THE CONTACT CONDITIONS

Following the basic idea of imposition of the contact conditions with non-conforming discretizations, described by París *et al.* (1995), compatibility equations will be imposed on one of the bodies, which will be called body A, whereas equilibrium equations will be imposed on the other body, called body B in what follows. In this way, displacements of body A will be a function of those of body B (i.e. the displacements of the points of the contact zone are defined by the displacements of the nodes of B), and the tractions of body B will be a function of those of body A (i.e. the tractions of the points of the contact zone are defined by the tractions of the nodes of A). Friction law will be imposed on the variables associated to the nodes of body A, which define contact stresses.

4.1. *Compatibility*

Two fields of displacements (12) and (13) will be defined, in order to apply compatibility conditions, on body A.

$$\Delta u_i^A(x), \quad \forall x \in D^A \Rightarrow \text{displacement solution of body } A \tag{12}$$

$$\Delta u_i^{\partial A}(y), \quad \forall y \in \partial D^A \Rightarrow \begin{cases} \Delta u_i^A(y) & \text{along } \partial D_f^A \\ -\Delta u_i^B(y) + \delta_i(y) & \text{along } \partial D_c^A \end{cases} \tag{13}$$

Notice that the displacement field $\Delta u_i^A(x)$, which corresponds to the solution of the problem, is defined at all points of body A, and will have associated a compatible strain field $\Delta \varepsilon_{ij}^A(x)$, whereas $\Delta u_i^{\partial A}(y)$ is only defined at points of the boundary. Note that the physical meaning of the variable $\delta_1(y)$ that appears in (13) is the value of how close or far the boundaries get at point y, whereas $\delta_2(y)$, also in (13), represents the relative displacement between the two surfaces at point y, both values with reference to an increment of external load.

To establish compatibility between these two fields of displacements, the Principle of Virtual Forces (i.e. the Principle of Virtual Work where the stress field is a virtual field in equilibrium and the displacement and strain fields correspond to the actual problem) is applied. This principle, for the case of absence of body forces, takes the expression:

$$\int_{D^A} \sigma_{ij}^{A\psi}(x)\, \Delta \varepsilon_{ij}^A(x)\, \mathrm{d}v = \int_{\partial D^A} t_i^{A\psi}(y)\, \Delta u_i^{\partial A}(y)\, \mathrm{d}s \tag{14}$$

an expression that must be satisfied for every virtual field of stresses $\sigma_{ij}^{A\psi}$ and $t_i^{A\psi}$ in equilibrium.

Transforming the first integral:

$$\begin{aligned} \int_{D^A} \sigma_{ij}^{A\psi}(x)\, \Delta \varepsilon_{ij}^A(x)\, \mathrm{d}v &= \int_{D^A} \sigma_{ij}^{A\psi}(x)\, \Delta u_{i,j}^A(x)\, \mathrm{d}v \\ &= \int_{D^A} (\sigma_{ij}^{A\psi}(x)\, \Delta u_i^A(x))_{,j}\, \mathrm{d}v - \int_{D^A} \sigma_{ij,j}^{A\psi}(x)\, \Delta u_i^A(x)\, \mathrm{d}v \\ &= \int_{\partial D^A} \sigma_{ij}^{A\psi}(y)\, \Delta u_i^A(y) n_j^A\, \mathrm{d}s \\ &= \int_{\partial D^A} t_i^{A\psi}(y)\, \Delta u_i^A(y)\, \mathrm{d}s \end{aligned} \tag{15}$$

where relation ε–u, the divergence theorem, internal equilibrium equations (with zero body forces) and the Cauchy lemma have been successively applied. Substituting (15) into (14) it follows that:

$$\int_{\partial D^A} t_i^{A\psi}(y)(\Delta u_i^A(y) - \Delta u_i^{\partial A}(y))\, \mathrm{d}s = 0 \tag{16}$$

Due to the fact that the fields of displacements Δu_i^A and $\Delta u_i^{\partial A}$, defined in (12) and (13), respectively, only differ along the contact zone, expression (16) leads to:

$$\int_{\partial D_c^A} t_i^{A\psi}(y)(\Delta u_i^A(y) + \Delta u_i^B(y) - \delta_i(y))\, \mathrm{d}s = 0 \tag{17}$$

an expression that must be fulfilled for any field $t_i^{A\psi}$ in equilibrium.

Expressing equation (17) in an approximate form, according to the discretization performed:

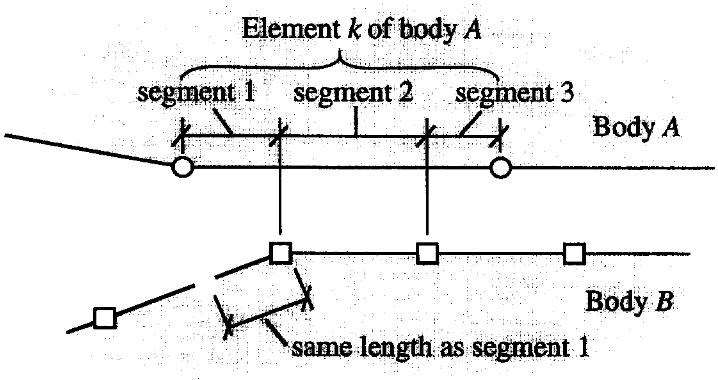


Fig. 4. Calculation scheme of integrals that appear in compatibility conditions.

$$\sum_{k}^{NCA} \int_{\partial D^{kA}} (\mathbf{N}^{kA}\mathbf{t}_i^{kA\psi})^{\mathrm{T}}(\mathbf{N}^{kA}\,\Delta\mathbf{u}_i^{kA} + \mathbf{N}^{B}\,\Delta\mathbf{u}_i^{B} - \mathbf{N}^{kA}\delta_i^{k})\,\mathrm{d}s = 0 \tag{18}$$

where NCA is the number of elements of the body A which belong to the potential contact zone and the superindex k makes reference to the element of NCA along which the integration is performed. Note that the shape function matrix $\mathbf{N}^B$ has been kept complete due to the fact that an element of A will not necessarily contact with a single element of B and vice versa.

Due to the fact that eqn (18) has to be fulfilled for any field $t_i^{A\psi}$ in equilibrium, the compatibility equation in discretized form takes the expression:

$$\sum_{k}^{NCA} \int_{\partial D^{kA}} (\mathbf{N}^{kA})^{\mathrm{T}}\mathbf{N}^{kA}\,\mathrm{d}s\,\Delta\mathbf{u}_i^{kA} + \sum_{k}^{NCA} \int_{\partial D^{kA}} (\mathbf{N}^{kA})^{\mathrm{T}}\mathbf{N}^{B}\,\mathrm{d}s\,\Delta\mathbf{u}_i^{B} - \sum_{k}^{NCA} \int_{\partial D^{kA}} (\mathbf{N}^{kA})^{\mathrm{T}}\mathbf{N}^{kA}\,\mathrm{d}s\delta_i^{k} = 0 \tag{19}$$

the dependency with respect to the integration variable having been omitted for the sake of simplicity.

The computation of the integrals that multiply to $\Delta\mathbf{u}_i^{kA}$ and δ_i^k is easy, due to the fact that they are integrals of products of shape functions, their values only being dependent on the length of the element k. With reference to those that multiply to $\Delta\mathbf{u}_i^{B}$, they will depend on the relative positions in which the nodes of B are being situated along the element k of A. Thus, a partition of the integral in segments defined by the positions of these nodes of B is required. These positions have been computationally determined by a mapping of the nodes of B on the elements of A, having taken a common curvilinear coordinate. Depending on the approach followed, this coordinate definition is maintained constant throughout the incremental process in a small displacement approach, or is updated at the beginning of each increment of load in a moderate or large displacement approach, where an updating of the geometry is required. This is illustrated in Fig. 4. An easy and direct way to evaluate the integrals along these segments is to use a Gauss–Legendre quadrature with two points along each of the segments, thus obtaining the exact value, according to the discretization performed, with linear elements, of the integral that appears in (19).

4.2. *Equilibrium*

Two stress states will be defined for the body B, analogously to what has been done for the displacements in Section 4.1 devoted to compatibility equations. Thus:

$$\Delta t_i^B(x,\mathbf{n}) = \Delta\sigma_{ij}^B(x)n_j, \quad \forall x \in D^B \Rightarrow \text{stress solution of body } B \tag{20}$$

$$\Delta t_i^{\partial B}(y), \quad \forall y \in \partial D^B \Rightarrow \begin{cases} \Delta t_i^B(y,\mathbf{n}^B) & \text{along } \partial D_f^B \\ \Delta t_i^A(y,\mathbf{n}^A) & \text{along } \partial D_c^B \end{cases} \tag{21}$$

where $\mathbf{n}$ is an arbitrary unit vector and $\mathbf{n}^K$ is the outward unit normal to the boundary of body K at point y.

Notice again that $\Delta t_i^B(x,\mathbf{n})$ is defined on the whole of body B, whereas $\Delta t_i^{\partial B}(y)$ is only defined along the boundary of body B.

Applying now the Principle of Virtual Displacements (i.e. the Principle of Virtual Work where the displacement and strain fields are virtual compatible fields and the stress field correspond to the actual problem), it must be fulfilled, in order to guarantee the equilibrium between these two stress fields, that:

$$\int_{D^B} \Delta\sigma_{ij}^B(x)\varepsilon_{ij}^{B\psi}(x)\,\mathrm{d}v = \int_{\partial D^B} \Delta t_i^{\partial B}(y)u_i^{B\psi}(y)\,\mathrm{d}s \tag{22}$$

for any field of displacements $u_i^{B\psi}$ and compatible strains $\varepsilon_{ij}^{B\psi}$.

Performing similar operations to those indicated for the compatibility equation in Section 4.1, the left hand side of eqn (22) can be transformed in the following way:

$$\begin{aligned} \int_{D^B} \Delta\sigma_{ij}^B(x)\varepsilon_{ij}^{B\psi}(x)\,\mathrm{d}v &= \int_{D^B} \Delta\sigma_{ij}^B(x)u_{i,j}^{B\psi}(x)\,\mathrm{d}v \\ &= \int_{D^B} (\Delta\sigma_{ij}^B(x)u_i^{B\psi}(x))_{,j}\,\mathrm{d}v - \int_{D^B} \Delta\sigma_{ij,j}^B(x)u_i^{B\psi}(x)\,\mathrm{d}v \\ &= \int_{\partial D^B} \Delta\sigma_{ij}^B(y)u_i^{B\psi}(y)n_j^B\,\mathrm{d}s \\ &= \int_{\partial D^B} \Delta t_i^B(y)u_i^{B\psi}(y)\,\mathrm{d}s \end{aligned} \tag{23}$$

which substituted into (22) leads to:

$$\int_{\partial D_c^B} (\Delta t_i^B(y) - \Delta t_i^A(y))u_i^{B\psi}(y)\,\mathrm{d}s = 0 \tag{24}$$

where the definition of $\Delta t_i^{\partial B}(y)$ given by (21) has been taken into account.

The equilibrium at all points y belonging to the contact zone, i.e. $\Delta t_i^A(y) = \Delta t_i^B(y)$, is guaranteed by the fulfilment of (24) for any fields $u_i^{B\psi}(y)$.

By a similar process to that followed in the obtention of (19) from (17), the following system of equations is obtained from (24):

$$\sum_k^{NCB} \int_{\partial D^{kB}} (\mathbf{N}^{kB})^{\mathrm{T}}\mathbf{N}^{kB}\,\mathrm{d}s\,\Delta\mathbf{t}_i^{kB} - \sum_k^{NCB} \int_{\partial D^{kB}} (\mathbf{N}^{kB})^{\mathrm{T}}\mathbf{N}^A\,\mathrm{d}s\,\Delta\mathbf{t}_i^A = 0 \tag{25}$$

The integrals that appear in these equations are similar to those that appeared in the compatibility equations and they are computed in a similar way.

It should be noticed that this manner of imposing equilibrium equations ensures the global equilibrium of forces and moments of the problem. The force equilibrium can be proved using in (24) [or in discretized form (25)] a constant displacement field $u_i^{B\psi}(y)$, which can be correctly described by linear elements. The moment equilibrium can be proved suitably using in (24) [or (25)] the components of the radius vector as displacement field $u_i^{B\psi}(y)$, which can also be described correctly by linear elements. Obviously, the finer the discretization used for $u_i^{B\psi}(y)$ is, the more similar the point values of tractions of the two bodies obtained from the system of equations (25) will be.

Similar comments to these for equilibrium equation (24), or in discretized form (25), may be applied to compatibility equation (17) or (19).

4.3. *Friction condition*

As has previously been stated, the Coulomb model will be used to describe friction effects.

- If during the application of an increment of load a pair of points in contact are in adhesion, the relative displacements between the two points is null and the condition to be imposed is:

$$\delta_2(y) = 0 \tag{26}$$

The limit of application of this condition is established by the maximum value allowable for the tangential component of the stress:

$$|t_2^A(y)| \leqslant \mu |t_1^A(y)| \tag{27}$$

- If during the application of an increment of load a pair of points in contact are sliding, the condition that must be imposed is the proportionality between the normal and tangential components of the stresses:

$$t_2^A(y) = \pm \mu t_1^A(y) \Rightarrow \Delta t_2^A(y) = \pm \mu \Delta t_1^A(y) \tag{28}$$

The sign + or − is selected depending on the straight line of sliding on which the stress vector is situated in Fig. 3. The limit of application of these conditions is fixed by the dissipative character of friction:

$$t_2^A(y)\delta_2(y) < 0 \tag{29}$$

5. THE ALGORITHM OF SOLUTION

To take into consideration the presence of friction, an incremental algorithm of solution must be employed. Then, given a set of compatible contact conditions at the beginning of an increment of load, these conditions are maintained constant during the application of the increment, the obtaining of the solution of the problem thus being guaranteed. The limit of application of these boundary conditions will produce the amount of load applicable at each increment, París and Garrido (1985, 1989). The detection of a compatible set of initial contact conditions for each increment (in particular the first) may require the application of a trial and error procedure.

There are two approaches for the generation of incremental algorithms: a displacement scaling approach, París and Garrido (1989) and a load scaling approach Man *et al.* (1992). París and Blázquez (1994) discussed the features of these two approaches, showing that they lead to the same results and that the displacement scaling approach involves less computational effort, for which reason it will be used in this paper and is briefly described in what follows.

Let us assume that after a certain application of load the i-th increment starts with a certain set of contact conditions. The system of equations to be solved in this increment of load is constituted by:

- The set of integral equations corresponding to the two bodies involved in the contact, eqn (11).

- Compatibility equations corresponding to the potential contact zone, applied to one of the bodies, called body A, eqn (19).
- Equilibrium equations corresponding to the potential contact zone, applied to the other body, called body B, eqn (25).
- The condition of free stress surface must be imposed on all the nodes n that belong to the potential contact zone of body A but are not in contact:

$$\Delta t_1^A(n) = \Delta t_2^A(n) = 0 \tag{30}$$

- If these nodes of body A are in contact, the condition:

$$\delta_1(n) = 0 \tag{31}$$

must be applied together with eqn (26) or (28) depending on whether the node is in adhesion or sliding respectively.

The remaining load is applied (in fact any value can be applied because what is interesting to detect is the evolution of the solution with the load), and once the system has been solved, all four possible limits of application of the contact conditions are checked, thus detecting the fraction of load λ_j $(0 \leqslant \lambda_j \leqslant 1, j = 1 \ldots 4)$ for which the end of linear behaviour for each type of condition is reached. The maximum admissible increment of load that can be applied is defined by the value $\lambda = \min(\lambda_j)$, these λ_j being associated to the following situations:

(1) Reduction of the contact zone, originated by the appearance of tractions at some node, n, of body A:

$$t_1^A(n)_{i-1} + \Delta t_1^A(n) > 0 \tag{32}$$

The correct value of λ_1 is calculated identifying the final value of the normal stress in (32) with zero.

$$t_1^A(n)_i = t_1^A(n)_{i-1} + \lambda_1 \, \Delta t_1^A(n) = 0 \Rightarrow \lambda_1 = -\frac{t_1^A(n)_{i-1}}{\Delta t_1^A(n)} \tag{33}$$

The node will be removed, after the application of this increment of load, from ∂D_c^A to ∂D_t^A. The situations described here can obviously arise at several nodes, the value of λ_1 defined by (33) being the minimum of all possible.

(2) When there are nodes of A that trespass the boundary of body B, this means an increase in the size of the contact zone. The fraction of load to be applied, λ_2, is calculated as that which collocates the trespassing node on the trespassed element of body B. Figure 5 illustrates, more clearly than a formula, París *et al.* (1995), the calculation of λ_2, which again must be the minimum of all possible.

After the application of the increment of load the node will pass from ∂D_t^A to ∂D_c^A.

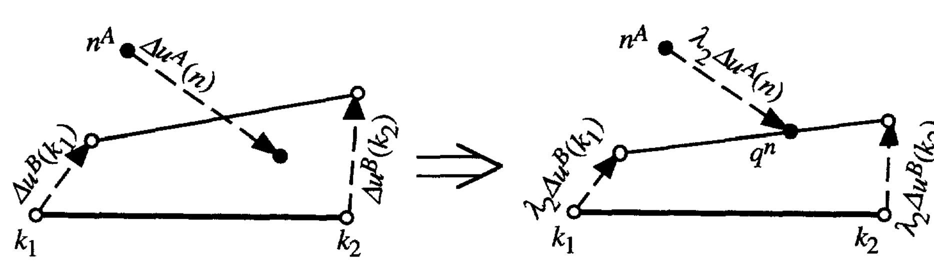


Fig. 5. Load factor for an increasing contact zone.

(3) When the dissipative character of the friction (29), is violated in a node that is sliding, the fraction of load is null, $\lambda_3 = 0$, the node being changed from ∂D^A_{cd} to ∂D^A_{ca}.
(4) When in a node in adhesion the tangential stress passes the limit given by the Coulomb friction law,

$$|t^A_2(n)_{i-1} + \Delta t^A_2(n)| > \mu |t^A_1(n)_{i-1} + \Delta t^A_1(n)| \tag{34}$$

the fraction of load to be applied is:

$$\lambda_4 = -\frac{-t^A_2(n)_{i-1} \pm \mu t^A_1(n)_{i-1}}{-\Delta t^A_2(n) \pm \mu \Delta t^A_1(n)} \tag{35}$$

The signs $\pm$ correspond to the two sliding lines, Fig. 2. After the application of the increment of load the node will be passed from ∂D^A_{ca} to ∂D^A_{cd}.

After an increment of load the geometry (and consequently the integration constants) may be updated, allowing the consideration of situations different from those covered by the small displacement theory.

6. RESULTS

Three classical problems belonging to the receding, conforming and advancing contact cases will be studied. All of them consist in the compression of a certain body against a rectangular foundation, the geometry of the body giving the different contact nature to the problem. These problems coincide with those analysed by other authors with different approaches, which facilitates comparison between them, this being of particular importance for verifying the improvements in the new formulation presented here. Due to the presence of symmetry of geometry and loads, only half of the problems will be discretized, an implicit scheme, París and Cañas (1997), being followed.

6.1. *Compression of a layer*

The complete problem and the one which will be analysed here, due to its symmetry, are shown, together with the properties of the layer and foundation, in Fig. 6.

The nature of the problem (combination of geometry and load) situates it in the receding contact problem class, the final size of the contact zone, independent in this case of the amount of load applied, being smaller than the initial. The problem, once the correct

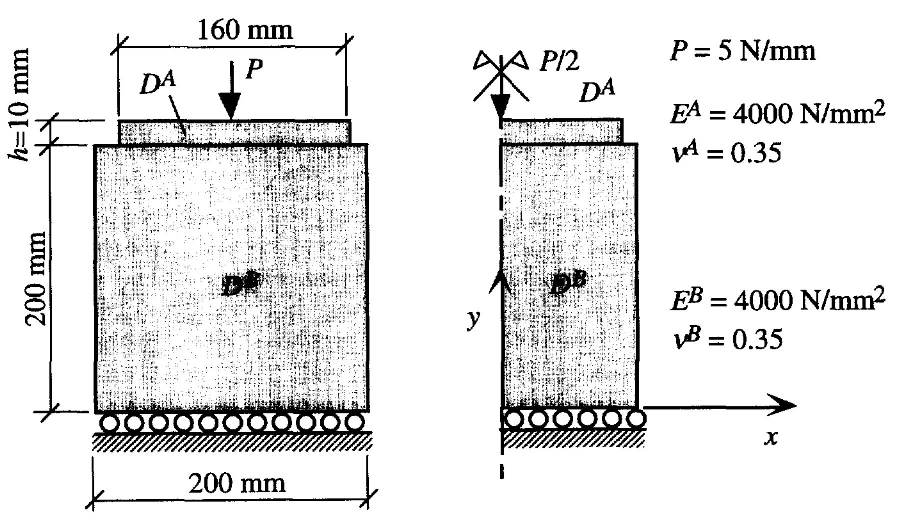


Fig. 6. Compression of a layer, geometry and properties.

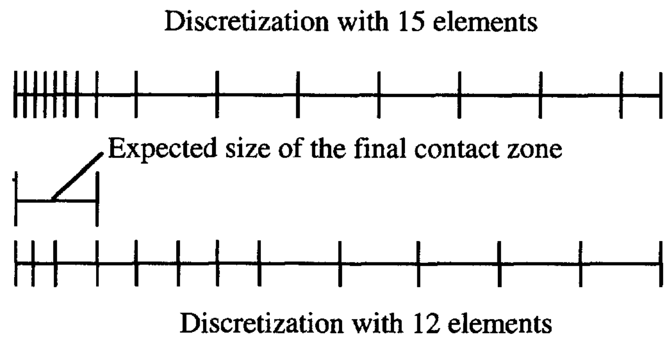


Fig. 7. Discretizations used along the initial contact zone for layer and foundation.

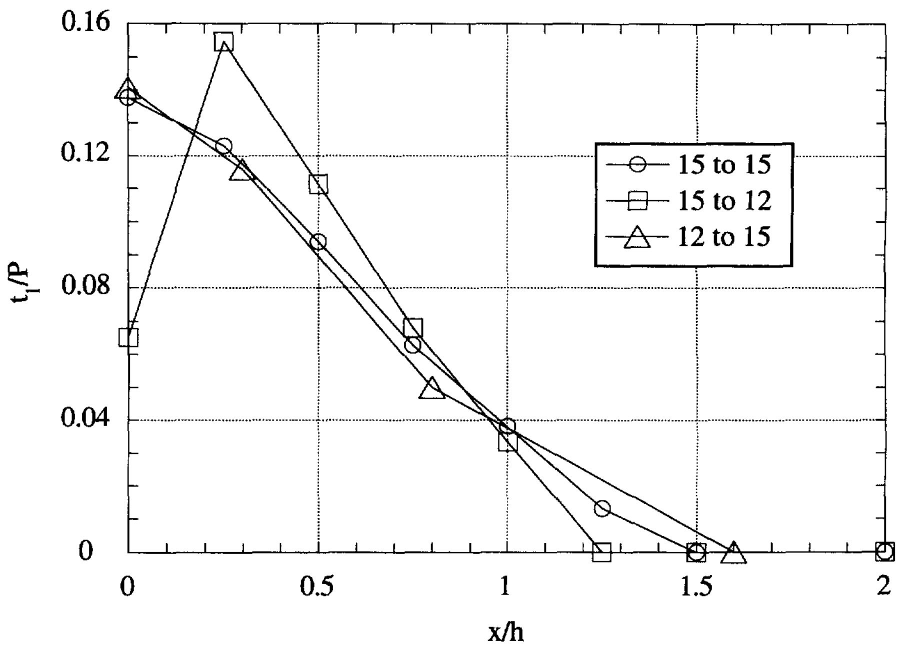


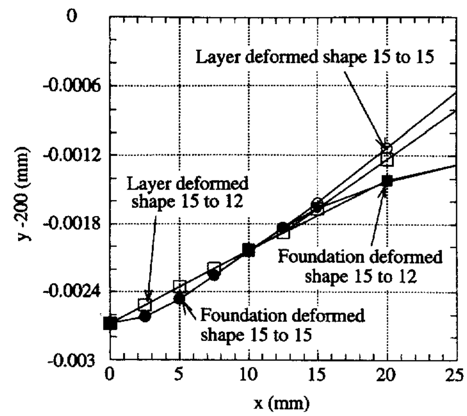
Fig. 8. Contact pressures for the frictionless case, $\mu = 0$.

size and partition of the contact zone have been detected by means of a trial and error procedure, is then solved in one increment of load, the finding of contact stresses presenting a smooth evolution along the contact zone being expected.

The two discretizations shown in Fig. 7, one having 12 and the other 15 elements along the potential (in this case the initial) contact zone, have been combined. Notice that they are two strongly non-conforming discretizations, particularly in what will be the final contact zone.

Figure 8 shows the distribution of contact stresses for the frictionless case for three combinations of discretizations. In all cases the layer has been taken as body A and the foundation as body B, the first number being associated to body A. This means that one conforming case is analysed (15–15), another where the body that controls the stresses is more finely discretized (15–12) and finally one case where the body that controls the displacements is more finely discretized (12–15).

It can be observed from Fig. 8 that whereas the case in which the body that controls the displacements is more finely discretized (case 12–15) presents similar results to the conforming cases (case 15–15), taken as reference, serious discrepancies with respect to both of them appear when the body that controls the stresses is better discretized (case 15–12), a clear jump in the contact pressure appearing in this last case. The origin of this jump



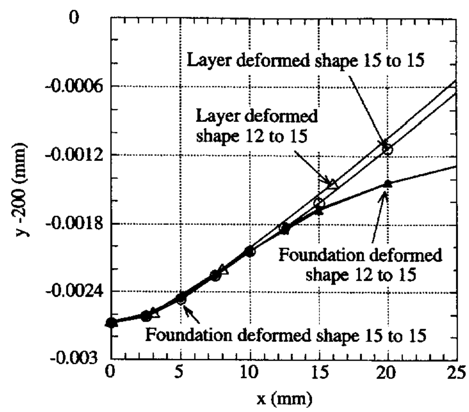


Fig. 9. Deformed configuration for the layer and the foundation for the frictionless case.

is qualitatively the same as the authors have already pointed to in previous papers employing non-conforming discretizations although based on a different philosophy of application of the contact conditions París *et al.* (1995), Blázquez *et al.* (1997). This is clearly explained inspecting the deformed configurations of the three cases, which are represented in Fig. 9.

The jumps in the contact stresses appear at nodes of body A which are forced, due to the discretization performed in body B, to be placed far from the natural position they ought to occupy in accordance with the discretization performed in body A. These jumps, similar to those obtained with a node-to-point approach, Blázquez *et al.* (1997), thus appear due to the over/under compression originated at a node to separate it from its natural position. This never happens when the body that controls the displacements is discretized more finely than the one that controls the stresses, a fact that prevents the appearance of jumps.

Although the previous discussion has referred to variables of the contact zone, the different discretization schemes affect all variables of the problem. Thus, in the example under consideration here the difference in the value of the gap at the extreme of the layer is 12% between the conforming case and the 15–12 case, this value being reduced to 1.3% in the 12–15 case.

The results associated to the case with presence of friction are shown in Fig. 10. The meshes, in order to have a more precise definition of the adhesion and sliding contact zone, have been refined (15 passes to 21 and 12 passes to 14), although maintaining their non relative conformity. For the reasons given above, only the conforming cases and the case where the body that controls the displacement is better discretized are considered, both presenting a similar behaviour.

6.2. *Compression of a rectangular punch*

The complete problem and the one which will be analysed here, due to its symmetry, are shown, together with the properties of the punch and the foundation, in Fig. 11. The nature of the problem (combination of geometry and load) makes it belong to the conforming contact problem class, the final size of the contact zone, independently of the amount of load applied, being identical to the initial. The problem, once the correct partition of the contact zone has been detected by means of a trial and error procedure, is then solved in an increment of load. The singular character of the stresses at the end of the contact zone is well known. This must obviously affect the degree of refinement of the discretizations performed for the two bodies in contact in this zone, independently of the non-conforming character of the approach employed here.

Two different discretizations, shown in Fig. 12, have been employed along the contact zone, both being identical along the rest of the boundaries. The coordinates of the nodes of the two discretizations are, to be specific, indicated in Table 1. Note the strong non-conforming character of the two discretizations close to the corner, which is the most delicate part of the problem, due to the above mentioned singular character of the stresses.

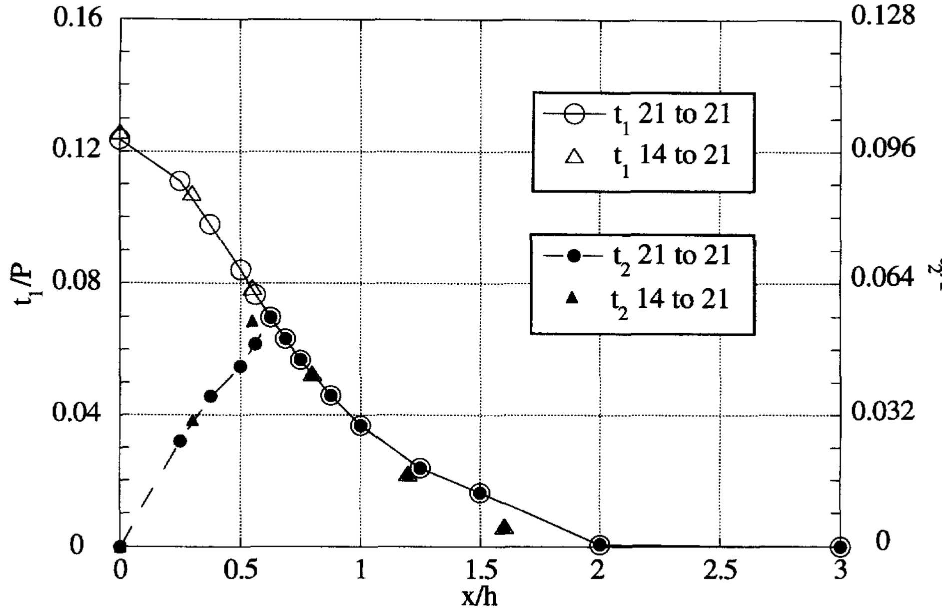


Fig. 10. Normal and tangential contact stresses for the case with friction, $\mu = 0.8$.

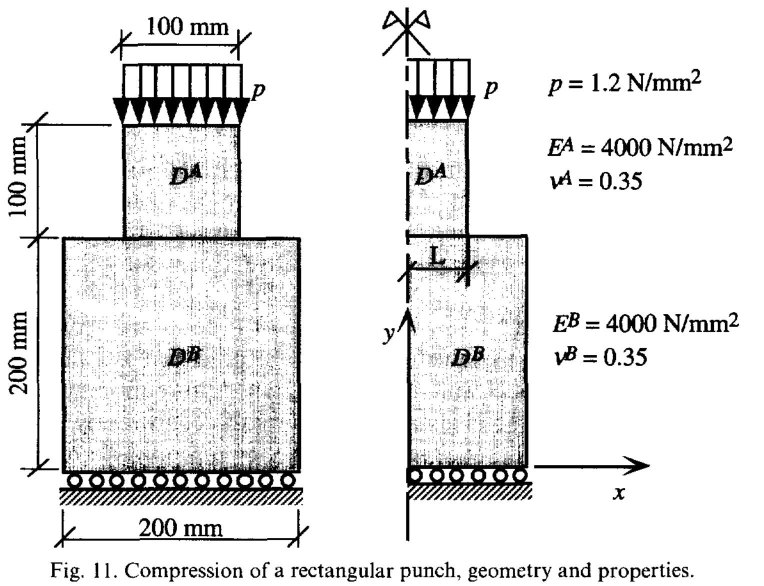


Fig. 11. Compression of a rectangular punch, geometry and properties.

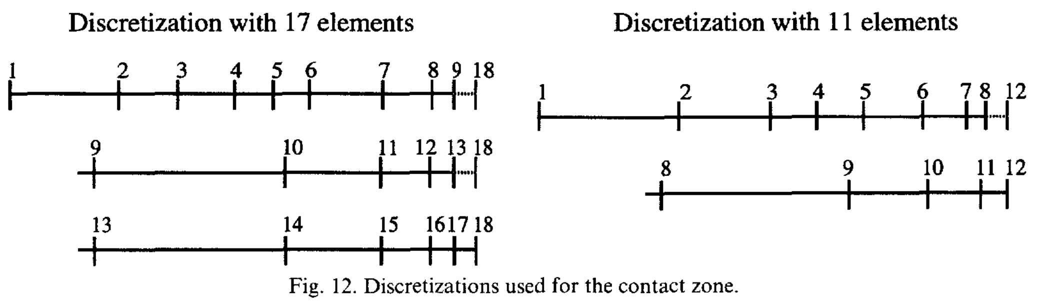


Fig. 12. Discretizations used for the contact zone.

Table 1. Coordinates x (mm) of the nodes of the contact zone for the two discretizations employed

17 elements	0 12 18 24 28 32 40 45 47.5 48.75 49.375 49.6875 49.84375 49.921875 49.9609375 49.98046875 49.990234375 50
11 elements	0 15 25 30 35 41 45.5 47.75 48.875 49.4375 49.7188 50

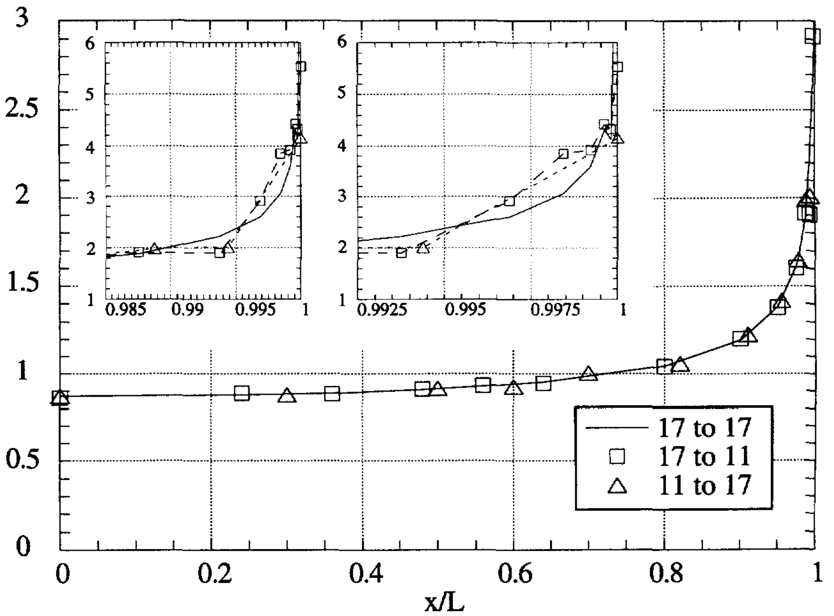


Fig. 13. Contact pressure in the punch in the frictionless case.

The punch has played the role of body A and the foundation the role of body B in all cases, the results being presented with the same notation as in the former example. Figures 13 and 14 present contact stresses that appear along the contact zone for the frictionless case and for a friction coefficient $\mu = 0.2$, respectively.

In the detail that appears in Fig. 13 can be clearly observed the presence of a jump at the element closest to the corner, in the case denoted 11–17, motivated by the presence of a singularity at the corner. This jump is due to the inability of the linear elements used to reproduce the presence of a singular field and in fact it also appears, in the corresponding position, for the case of conforming discretization, denoted by 17–17, and for the non-conforming 17–11 case. However, the most apparent jumps in the case denoted 17–11 appear several nodes from the corner. These jumps arise because the displacements of the foundation (body B), which has fewer nodes in the contact zone, force the nodes of the punch (body A) to occupy positions different to those they naturally ought to occupy.

Note that all these facts can be observed in the case with friction, Fig. 14, where some of them can be seen even more clearly.

Thus, the results show the same tendencies as in the former example: when modelling more finely the body that controls the displacements the results are as satisfactory as with conforming discretizations, whereas when modelling with greater refinement the body that controls the stresses some jumps may appear. It must be pointed out that this general rule is applicable even in a case, like the one analysed here with presence of singularities, where the natural tendency would be to put more elements in the body that controls the stresses. In any case, if the difference in the discretizations is not too great, as is the case far from the corner, the solution is very similar to that obtained with the conforming discretization.

6.3. *Compression of a cylinder*

The complete problem and the one which will be analysed here, due to its symmetry, are shown, together with the properties of the cylinder and the foundation, in Fig. 15. The

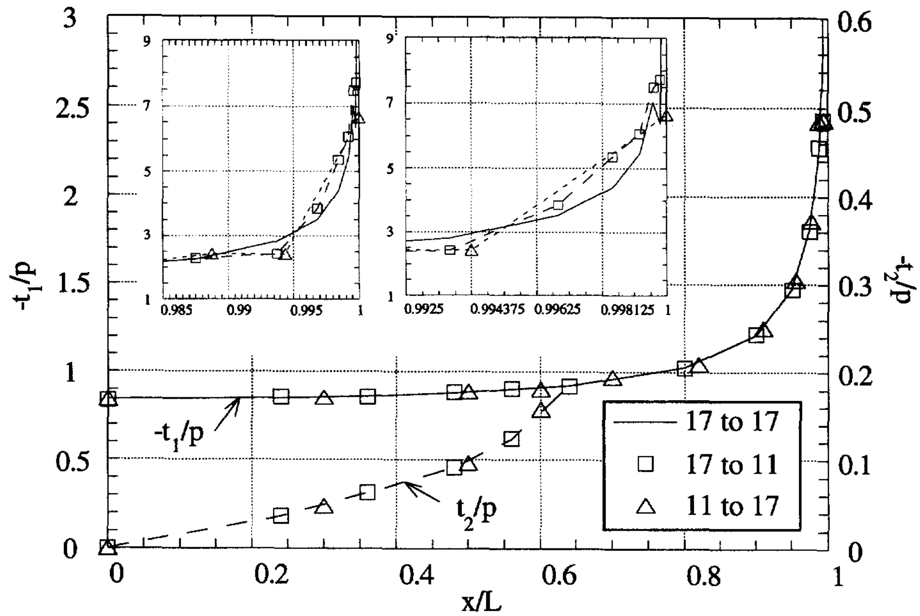


Fig. 14. Normal and tangential stresses in the punch with friction, $\mu = 0.2$.

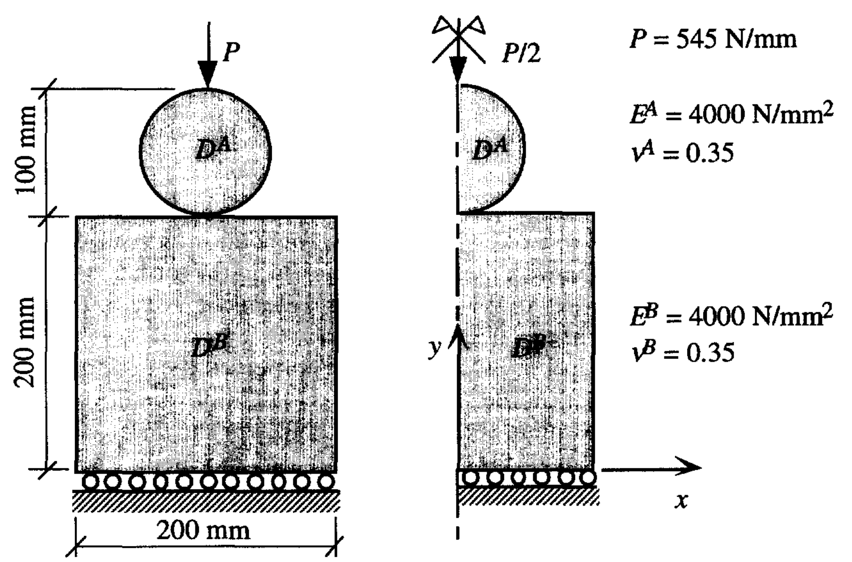


Fig. 15. Compression of a cylinder, geometry and properties.

nature of the problem makes this problem belong to the advancing contact problem class, the size of the contact zone increasing progressively as the amount of load increases. The problem must necessarily be solved incrementally.

Discretizations with 11, 12 and 13 elements uniformly distributed along the contact zone, which estimated according to Hertz theory has a size of approximately $s_{max} = 4$ mm, have been used. In all cases analysed the cylinder has played the role of body A and the foundation the role of body B. The notation used to denote the different cases is similar to that used in the former cases.

The results obtained for the problem without friction are compared with the analytical solution obtained by Hertz, Fig. 16, a very satisfactory agreement among all the combinations of discretizations being observed, although it could be argued that in this case the degree of non-conformity of the two discretizations is not too great.

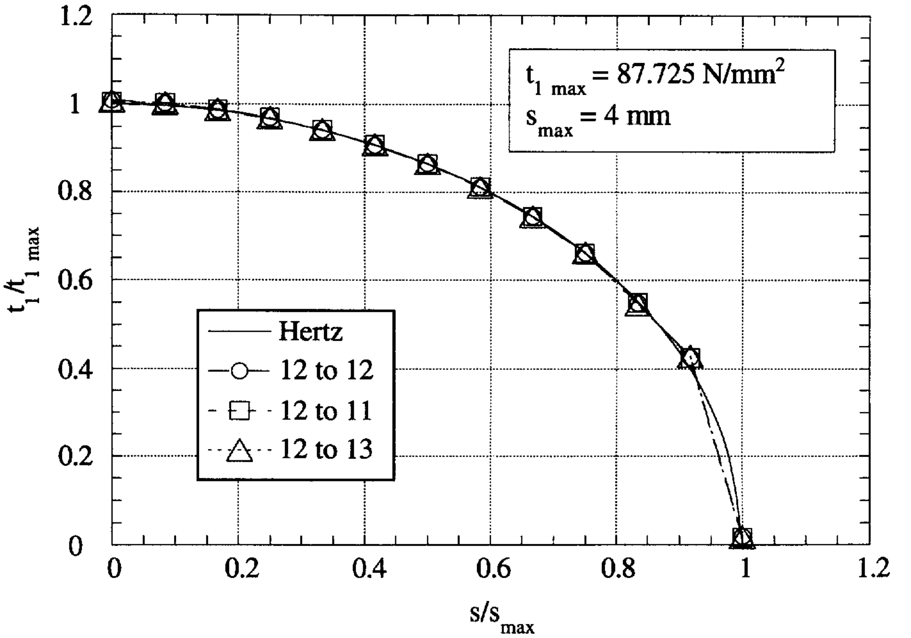


Fig. 16. Contact pressures for the frictionless case.

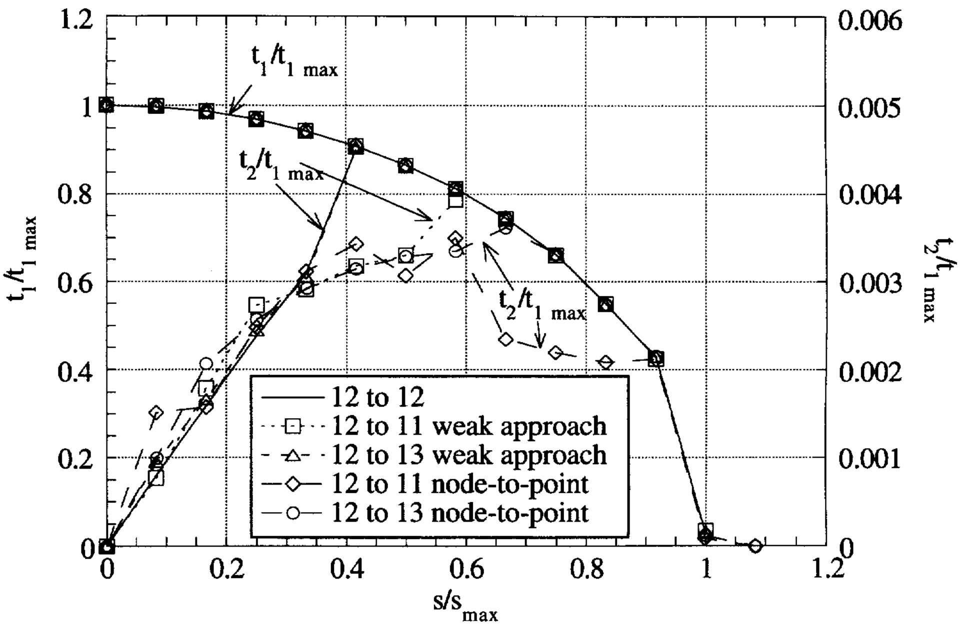


Fig. 17. Normal and tangential stresses along the contact zone with a friction coefficient $\mu = 0.005$.

The results corresponding to the same case with a friction coefficient $\mu = 0.005$ are shown in Fig. 17. In order to have a clear idea of the benefit of the weak application, results obtained with a node-to-point scheme, Blázquez *et al.* (1997), have also been presented.

The weak approach with more elements representing the body that controls displacements (12–13) produces almost the same results as the conforming case taken as reference. As usual, the results are slightly poorer when more elements are used for the body controlling stresses (12–11). It might be argued that these excellent results for the case where the body that controls the displacements is more finely represented could be due to the apparently low degree of non-conformity between the discretizations (12–13), but the results became clearly much poorer when a node-to-point contact scheme was used, with

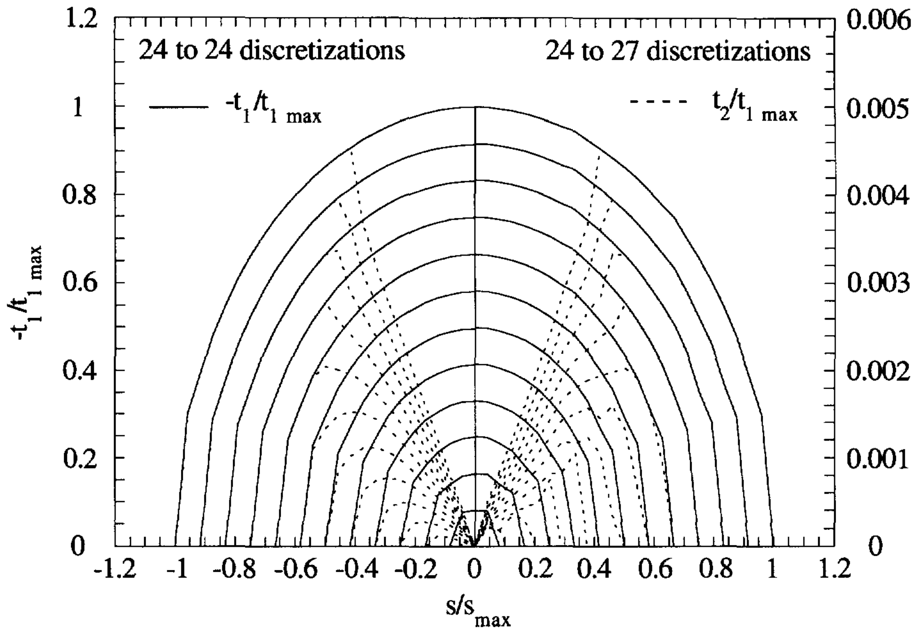


Fig. 18. Evolution of the stresses with the load for conforming and non-conforming discretizations, with a friction coefficient $\mu = 0.005$.

the same influence of the relative discretizations of the bodies controlling displacements and stresses. In Blázquez *et al.* (1997) it was shown how discretizations similar to this, using other node-to-point contact schemes, originate very serious jumps in the stresses, which focuses the real reason for the results obtained here on the manner of applying contact conditions.

The consistency of the procedure can be appreciated in the results shown in Fig. 18, where the evolution of the contact stresses with the load is shown for more refined conforming and non-conforming discretizations. No significant jumps appear.

7. CONCLUSIONS

A new approach in the application of the contact conditions for the case of non-conforming discretizations with boundary elements of the surfaces involved in the contact has been presented. Previous formulations satisfied equilibrium and compatibility locally, identifying values of displacements and tractions at nodes of one of the bodies with the corresponding contacting point of the other body. In the new approach, compatibility of displacements and equilibrium of tractions are applied globally by means of variational principles, the Principles of Virtual Displacements and Virtual Forces.

The aim of the development of a new approach has been to eliminate the jumps (oscillations) in the tractions that the authors had detected as possibly appearing, using any of the variants of a general node-to-point approach. The new approach, which relaxes locally equilibrium and compatibility requirements, is not as rigid as the former one in the application of the contact conditions and no jumps have been detected in the problems tested.

What is still necessary, as a general rule, is to discretize more finely (in fact at least no less finely), the body that controls the displacements. This restriction is inherent in a non-conforming discretization approach and, as has already been studied by the authors in previous papers París *et al.* (1995) and Blázquez *et al.* (1997), it tends to avoid a poor definition of the displacements at the nodes of the body that controls the stresses, subsequently forcing those nodes to occupy positions different from those they would tend to occupy and thus slightly altering the value of the traction at this point. Due to the fact that the assignment of bodies A or B to the two bodies involved in a contact zone is arbitrary, the former restriction does not represent any complication or limitation. The discretizations can be performed freely and the program will assign the role of body B to the one that has been discretized with more elements.

The new approach is obviously more complicated from a computational point of view, both as regards programming and computation time. However, the results obtained with three classical problems belonging to the receding, conforming and advancing contact categories, with absence of nonsense jumps in the contact stresses, support the robustness of the procedure and its use in a non-conforming discretization approach.

It is important to note that the approach presented is valid with no modifications for small, moderate and large displacements whose only difference in treatment, assuming linear behaviour of the materials involved in the contact problem, is the necessity or otherwise of updating the geometry after the application of the increments of load.

Although the approach presented in this paper has been related to contact problems, it would also be immediately applicable to substructuring techniques, allowing independent discretizations (of the interface) to be used for each substructure.

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